

# PARITY PRESERVATION FOR $K$ -TYPES IN AN IRREDUCIBLE REPRESENTATION

XIANG FAN

**ABSTRACT.** A phenomenon of parity preservation for  $K$ -types occurring in an irreducible admissible representation happens for various reductive Lie groups. This note gives a uniform proof of this fact for members of real reductive dual pairs, by applying Howe's duality theory of theta correspondence.

## 1. INTRODUCTION

To study a continuous representation  $(\pi, V)$  of a reductive Lie group  $G$  with a maximal compact subgroup  $K$ , we consider its  $K$ -spectrum, namely, the  $K$ -module decomposition of the subspace  $V_K$  of  $K$ -finite smooth vectors in  $V$ :

$$V_K|_K \simeq \bigoplus_{\sigma \in \mathcal{R}(K)} W_\sigma^{\oplus m(\sigma, \pi)},$$

where  $\mathcal{R}(K)$  is the set of  $K$ -types (i.e., equivalence classes of irreducible representations of  $K$ ),  $W_\sigma$  an underlying space of a  $K$ -type  $\sigma$ , and  $m(\sigma, \pi) = \dim \operatorname{Hom}_K(\sigma, \pi)$  the multiplicity of  $\sigma$  in  $\pi$ . We say that  $\sigma$  occurs in  $\pi$  if  $m(\sigma, \pi) > 0$ . Then

$$\mathcal{R}(K, \pi) = \{\sigma \in \mathcal{R}(K) \mid \operatorname{Hom}_K(\sigma, \pi) \neq 0\}$$

is the set of  $K$ -types occurring in  $\pi$ . We call  $\pi$  admissible if  $m(\sigma, \pi) < \infty$  for all  $\sigma \in \mathcal{R}(K)$ . The *admissible dual*  $\mathcal{R}(G)$  (the set of infinitesimal equivalence classes of irreducible admissible representations) of  $G$  is crucial in representation theory.

An interesting parity preservation phenomenon holds for  $K$ -spectrums of irreducible admissible representations of various  $G$ : for any given  $\pi \in \mathcal{R}(G)$ ,  $K$ -types in  $\mathcal{R}(K, \pi)$  have the same *parity*. Here the “parity” means a function  $\varepsilon : \mathcal{R}(K) \rightarrow \mathbb{Z}/2\mathbb{Z}$  such that  $\varepsilon(\sigma)$  is the parity of the sum of  $\mathbb{Z}$ -coefficients parametrizing the highest weight of a  $K$ -type  $\sigma$  (amended with more information if  $K$  is disconnected). This phenomenon can be well illustrated by the baby example of  $G = SL_2(\mathbb{R})$ ; nevertheless, this note aims to generalize it for more  $G$ .

**Example 1.1.** Let  $G = SL_2(\mathbb{R})$  with  $K = \left\{ k(\theta) = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \mid \theta \in \mathbb{R} \right\}$ .

Now  $\mathcal{R}(K) = \{\tau_m \mid m \in \mathbb{Z}\}$ , with  $\tau_m$  the character  $\tau_m(k(\theta)) = e^{im\theta}$ . Let  $\pi \in \mathcal{R}(G)$  with an underlying space  $V$ , and  $V(m) = \{v \in V \mid \pi(k)v = \tau_m(k)v, \text{ for any } k \in K\}$  for  $m \in \mathbb{Z}$ . Then  $\{m \in \mathbb{Z} \mid V(m) \neq 0\}$  is a subset of either  $2\mathbb{Z}$  or  $1 + 2\mathbb{Z}$ .

*Proof.* Let  $H = -i \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ ,  $R = \frac{1}{2} \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}$ , and  $L = \frac{1}{2} \begin{bmatrix} 1 & -i \\ -i & -1 \end{bmatrix}$ . Then  $[H, R] = 2R$ ,  $[H, L] = -2L$ ,  $[R, L] = H$ , and  $V(m) = \{v \in V_K \mid \pi(H)v = mv\}$ .

2000 *Mathematics Subject Classification.* Primary 22E46, 11F27.

*Key words and phrases.*  $K$ -types, reductive dual pair, Howe duality, theta correspondence.

Take a non-zero  $v_0 \in V(m_0) \neq 0$ , then  $\{\pi(R^{j_1} H^{j_2} L^{j_3})v_0 \mid j_i \in \mathbb{Z}_{\geq 0}\}$  spans  $V_K$ , and  $\pi(R^{j_1} H^{j_2} L^{j_3})v_0 \in V(m_0 + 2j_1 - 2j_3)$ . So  $\{m \in \mathbb{Z} \mid V(m) \neq 0\} \subseteq m_0 + 2\mathbb{Z}$ .  $\square$

To state our results, we clarify the “parity”  $\varepsilon : \mathcal{R}(K) \rightarrow \mathbb{Z}/2\mathbb{Z}$  in terms of explicit parametrizations. We will encounter  $U(n)$ -types,  $Sp(n)$ -types and  $O(n)$ -types.

For  $K = U(n)$ , take its maximal torus  $T = U(1)^n = \text{diag}(U(1), \dots, U(1))$ , and the positive system of roots  $\Delta^+(\mathfrak{t}, \mathfrak{k}) = \{e_i - e_j \mid 1 \leq i < j \leq n\}$ . Write each weight in  $\mathfrak{it}_0^*$  as the  $n$ -tuple of coefficients under the basis  $e_1, \dots, e_n$ . Then a  $U(n)$ -type is parametrized by its *highest weight*  $(a_1, \dots, a_n)$  with integers  $a_1 \geq a_2 \geq \dots \geq a_n$ .

For  $K = Sp(n)$ , take its maximal torus  $T = Sp(1)^n = \text{diag}(Sp(1), \dots, Sp(1))$ , and the positive system of roots  $\Delta^+(\mathfrak{t}, \mathfrak{k}) = \{e_i \pm e_j \mid 1 \leq i < j \leq n\} \cup \{2e_i \mid 1 \leq i \leq n\}$ . Write each weight in  $\mathfrak{it}_0^*$  as the  $n$ -tuple of coefficients under the basis  $e_1, \dots, e_n$ . Then a  $Sp(n)$ -type is parametrized by its *highest weight*  $(a_1, \dots, a_n)$  with integers  $a_1 \geq a_2 \geq \dots \geq a_n \geq 0$ .

For disconnected  $O(n)$ , the parametrization is a little complicated.

**Lemma 1.2** ([Wey39]). *Note that  $O(n) = U(n) \cap GL(n, \mathbb{R})$ . There is a bijection*  

$$\mathcal{R}(O(n)) \xrightarrow{\sim} \{(b_1, b_2, \dots, b_r, \underbrace{1, \dots, 1}_s, \underbrace{0, \dots, 0}_{n-r-s}) \in \mathcal{R}(U(n)) \mid 2r + s \leq n, b_r \geq 2\}$$

*such that an  $O(p)$ -type  $\sigma$  corresponds to the unique  $U(p)$ -type  $\lambda$  such that the highest weight vectors of  $\lambda$  generate an  $O(p)$ -module equivalent to  $\sigma$ . Therefore, an  $O(p)$ -type  $\sigma$  can be parametrized as:*

| if $r + s \leq \frac{n}{2}$  | if $\frac{n}{2} \leq r + s \leq n - r$   |
|--|--|
| $(b_1, b_2, \dots, b_r, \underbrace{1, \dots, 1}_s, \underbrace{0, \dots, 0}_{[\frac{n}{2}] - r - s}; +1)$ | $(b_1, b_2, \dots, b_r, \underbrace{1, \dots, 1}_s, \underbrace{0, \dots, 0}_{[\frac{n}{2}] - n + r + s}; -1)$ |

*Remark.* When  $r + s = \frac{n}{2}$ , these two cases coincide and give the same  $\sigma$ .

For  $K = U(n)$ ,  $Sp(n)$  or  $O(n)$ , define the parity  $\varepsilon : \mathcal{R}(K) \rightarrow \mathbb{Z}/2\mathbb{Z}$  explicitly as:

| $K$     | a $K$ -type $\sigma$ is parametrized as  | parity $\varepsilon(\sigma) \in \mathbb{Z}/2\mathbb{Z}$                                |
|---------|--|--|
| $U(n)$  | $(a_1, \dots, a_n)$ with integers $a_1 \geq a_2 \geq \dots \geq a_n$   | $\sum_{i=1}^n a_i \pmod{2}$  |
| $Sp(n)$ | $(a_1, \dots, a_n)$ with integers $a_1 \geq a_2 \geq \dots \geq a_n \geq 0$  | $\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} a_i + \frac{1-\epsilon}{2} \cdot n \pmod{2}$ |
| $O(n)$  | $(a_1, \dots, a_{[\frac{n}{2}]}; \epsilon)$ with integers $a_1 \geq a_2 \geq \dots \geq a_{[\frac{n}{2}]} \geq 0$ , and $\epsilon \in \{\pm 1\}$ | $\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} a_i + \frac{1-\epsilon}{2} \cdot n \pmod{2}$ |

For  $K = K_1 \times \dots \times K_r$  with  $K_i = U(n_i)$ ,  $Sp(n_i)$  or  $O(n_i)$ , as  $\mathcal{R}(K) = \{\bigotimes_{i=1}^r \sigma_i \mid \sigma_i \in \mathcal{R}(K_i)\}$ , define the parity  $\varepsilon : \mathcal{R}(K) \rightarrow \mathbb{Z}/2\mathbb{Z}$  by  $\varepsilon(\bigotimes_{i=1}^r \sigma_i) = \sum_{i=1}^r \varepsilon(\sigma_i)$ .

The main result of this note can be stated as the following theorem.

**Theorem 1.3** (Parity preservation). *Let  $G$  and  $K$  be as in the following table, with  $K$  embedded in  $G$  as a maximal compact subgroup in the usual way.*

| $G$                | $K$     | $G$               | $K$                  | $G$                   | $K$     |
|--------------------|---------|-------------------|----------------------|-----------------------|---------|
| $GL_m(\mathbb{R})$ | $O(m)$  | $O_p(\mathbb{C})$ | $O(p)$               | $Sp_{2n}(\mathbb{C})$ | $Sp(n)$ |
| $GL_m(\mathbb{C})$ | $U(m)$  | $O(p, q)$         | $O(p) \times O(q)$   | $Sp_{2n}(\mathbb{R})$ | $U(n)$  |
| $GL_m(\mathbb{H})$ | $Sp(m)$ | $Sp(p, q)$        | $Sp(p) \times Sp(q)$ | $O^*(2n)$             | $U(n)$  |
|                    |         | $U(p, q)$         | $U(p) \times U(q)$   |                       |         |

If  $\pi \in \mathcal{R}(G)$  and  $\sigma_1, \sigma_2 \in \mathcal{R}(K, \pi)$ , then  $\varepsilon(\sigma_1) = \varepsilon(\sigma_2)$ .

Theorem 1.3 seems approachable by a tedious and cases-by-case calculation of the multiplicities of  $K$ -types in parabolic inductions by the Frobenius reciprocity.

However, we will give a uniform and concise proof by applying Howe's duality theory of theta correspondence. It is based on the degree-parity preservation Theorem 2.5 for theta correspondence, which may be well-known to experts. Our observation is that Theorem 2.5, and the non-vanishing results for theta liftings in stable range (Lemma 3.1) or of type II (Lemma 3.2), together with the fact that the parity  $\varepsilon$  can be realized as the degree-parity for suitable dual pairs (Lemma 5.2), imply the parity preservation Theorem 1.3 for members of real reductive dual pairs.

## 2. DEGREE-PARITY PRESERVATION

In this section, we briefly review Howe's theory [How89] of *duality correspondence* (or *theta correspondence*), and prove a degree-parity preservation Theorem 2.5.

A real *reductive dual pair* is a pair  $(G, G')$  of closed reductive subgroups of  $\mathbf{Sp} = Sp_{2N}(\mathbb{R})$  (for some  $N$ ) such that they are mutual centralizers of each other. For a subgroup  $E \subseteq \mathbf{Sp}$ , write  $\tilde{E}$  for its preimage in the *metaplectic cover* (the unique non-trivial two-fold central extension)  $\tilde{\mathbf{Sp}}$  of  $\mathbf{Sp}$ . Take the Segal-Shale-Weil *oscillator representation* (c.f. [Sha62, Wei64, LV80])  $\omega$  of  $\tilde{\mathbf{Sp}}$  (with respect to the character of  $\mathbb{R}$  that sends  $t$  to  $e^{2\pi it}$ ), with associated smooth representation  $\omega^\infty$ .

**Theorem 2.1** (Howe duality, theta correspondence [How89]). *For a real reductive dual pair  $(G, G')$ , if  $\pi \in \mathcal{R}(\tilde{G})$ , then*

$$\omega^\infty \Big/ \bigcap_{T \in \text{Hom}_{\tilde{G}}(\omega^\infty, \pi)} \text{Ker}(T) \simeq \pi \otimes \Theta(\pi),$$

where  $\Theta(\pi)$  is a finitely generated admissible representation of  $\tilde{G}'$  with a unique irreducible quotient  $\theta(\pi)$  called the “theta lift” of  $\pi$ .

- Here “ $\otimes$ ” is not the usual tensor product; however, it is got from the tensor product of Harish-Chandra modules.
- The uniqueness means that the maximal proper sub- $\tilde{G}'$ -module of  $\Theta(\pi)$  is unique, and it is the kernel of  $\Theta(\pi) \rightarrow \theta(\pi)$ .

Let  $\mathfrak{sp} = \mathfrak{sp}(2N, \mathbb{C})$  denote the complexified Lie algebra of  $\mathbf{Sp}$ , and take  $\mathbf{U} = U(N)$  as a maximal compact subgroup of  $\mathbf{Sp}$ . We assume that  $K = \mathbf{U} \cap G$  and  $K' = \mathbf{U} \cap G'$  are maximal compact subgroups of  $G$  and  $G'$  respectively. Fock model realizes the  $(\mathfrak{sp}, \tilde{\mathbf{U}})$ -module of  $\omega$  as  $\mathcal{P} = \text{Poly}(\mathbb{C}^n)$ , the space of polynomials on  $\mathbb{C}^n$ . Note a decomposition  $\mathcal{P} = \mathcal{P}_0 \oplus \mathcal{P}_1$  as  $(\mathfrak{sp}, \tilde{\mathbf{U}})$ -modules, where  $\mathcal{P}_i$  is the linear span of all homogeneous polynomials in  $\mathcal{P}$  with degree  $\equiv i \pmod{2}$ . This gives rise to a decomposition of  $\tilde{\mathbf{Sp}}$ -modules  $\omega = \omega_0 \oplus \omega_1$ , where  $\omega_i$  has  $(\mathfrak{sp}, \tilde{\mathbf{U}})$ -module  $\mathcal{P}_i$ . Then as  $\tilde{G}'$ -modules  $\Theta(\pi) = \Theta_0(\pi) \oplus \Theta_1(\pi)$  with  $\omega_i^\infty \Big/ \bigcap_{T \in \text{Hom}_{\tilde{G}}(\omega_i^\infty, \pi)} \text{Ker}(T) \simeq \pi \otimes \Theta_i(\pi)$ .

**Corollary 2.2.**  $\Theta_i(\pi) = 0$  for some  $i \in \{0, 1\}$ . Therefore,  $\text{Hom}_{\tilde{G}}(\omega_i^\infty, \pi) = 0$ .

*Proof.* Otherwise,  $\Theta(\pi) = \Theta_0(\pi) \oplus \Theta_1(\pi)$  with both  $\Theta_i(\pi)$  non-zero finitely generated admissible  $\tilde{G}'$ -modules. By Zorn's Lemma,  $\Theta_i(\pi)$  contains a maximal proper submodule  $\Pi_i$ . Then  $\Theta_i(\pi)/\Pi_i$  is a non-zero irreducible admissible  $\tilde{G}'$ -module. For both  $i \in \{0, 1\}$ , the compositions of natural projections  $\Theta(\pi) \rightarrow \Theta_i(\pi) \rightarrow \Theta_i(\pi)/\Pi_i$  are linear, surjective and  $\tilde{G}'$ -intertwining, in contradiction with the uniqueness of the irreducible quotient of  $\Theta(\pi)$  in Theorem 2.1.  $\square$

**Proposition 2.3.** For  $i \in \{0, 1\}$ , if  $\sigma \in \mathcal{R}(\tilde{K}, \mathcal{P}_i)$ , then  $\text{Hom}_{\tilde{K}}(\omega_i^\infty, \sigma) \neq 0$ .

*Proof.* Take  $0 \neq \phi \in \text{Hom}(W_\sigma, \mathcal{P}_i)$ , with  $W_\sigma$  an underlying space of  $\sigma$ . Then  $\phi(W_\sigma)$  is a sub- $\tilde{K}$ -module of  $\omega_i$ , and  $(\omega_i|_{\tilde{K}}, \phi(W_\sigma)) \simeq (\sigma, W_\sigma)$  by Schur's lemma. The unitary representation  $\omega_i$  acts on a Hilbert space  $H$ , in which the finite-dimensional subspace  $\phi(W_\sigma)$  is closed, so  $H = \phi(W_\sigma) \oplus \phi(W_\sigma)^\perp$ . This decomposition gives a  $\tilde{K}$ -intertwining projection  $p : H \rightarrow \phi(W_\sigma)$ . The underlying space  $H^\infty$  of  $\omega_i^\infty$  is dense in  $H$ , so  $p|_{H^\infty} \neq 0$  and gives rise to a non-zero element of  $\text{Hom}_{\tilde{K}}(\omega_i^\infty, \sigma)$ .  $\square$

**Corollary 2.4.**  $\mathcal{R}(\tilde{K}, \mathcal{P}_0) \cap \mathcal{R}(\tilde{K}, \mathcal{P}_1) = \emptyset$ .

*Proof.* Let  $M$  be the centralizer of  $K$  in  $\mathbf{Sp}$ , then  $(K, M)$  is also a reductive dual pair [How89, Fact 1]. For the dual pair  $(K, M)$  and any  $\tilde{K}$ -type  $\sigma$ , Corollary 2.2 asserts that  $\text{Hom}_{\tilde{K}}(\omega_i^\infty, \sigma) = 0$  for some  $i \in \{0, 1\}$ . By Proposition 2.3,  $\sigma \notin \mathcal{R}(\tilde{K}, \mathcal{P}_i)$ .  $\square$

For  $\sigma \in \mathcal{R}(\tilde{K}, \mathcal{P})$ , [How89] defines the “degree”  $\deg(\sigma)$  (with respect to the dual pair  $(G, G')$ ) as the minimal degree of polynomials in the  $\sigma$ -isotypic subspace  $\mathcal{P}_\sigma = \sum_{\varphi \in \text{Hom}_{\tilde{K}}(\sigma, \mathcal{P})} \text{Im}(\varphi)$ .

**Theorem 2.5.** For a real reductive dual pair  $(G, G')$ , if  $\pi \in \mathcal{R}(\tilde{G})$  with  $\theta(\pi) \neq 0$ , and  $\sigma_1, \sigma_2 \in \mathcal{R}(\tilde{K}, \pi)$ , then  $\deg(\sigma_1) \equiv \deg(\sigma_2) \pmod{2}$ .

*Proof.* By definition,  $\text{Hom}_{\tilde{G}}(\omega^\infty, \pi) \neq 0$ . By Corollary 2.2,  $\text{Hom}_{\tilde{G}}(\omega_i^\infty, \pi) \neq 0$  for exactly one  $i \in \{0, 1\}$ . Take  $0 \neq T \in \text{Hom}_{\tilde{G}}(\omega_i^\infty, \pi)$ . Let  $V$  be the associated  $(\mathfrak{g}, \tilde{K})$ -module of  $\pi$ , consisting of  $\tilde{K}$ -finite vectors. Note that  $\mathcal{P}_i$  is dense in  $\omega_i^\infty$ , and vectors in  $\mathcal{P}_i$  are  $\tilde{K}$ -finite, so  $0 \neq T|_{\mathcal{P}_i} \in \text{Hom}_{\mathfrak{g}, \tilde{K}}(\mathcal{P}_i, V)$ . As a  $(\mathfrak{g}, \tilde{K})$ -module  $V$  is irreducible, so  $T(\mathcal{P}_i) = V$ .

For both  $j \in \{0, 1\}$ , let  $W_{\sigma_j}$  be an underlying space of  $\sigma_j$ , then  $\text{Hom}_{\tilde{K}}(W_{\sigma_j}, V) \neq 0$ . Take  $0 \neq \varphi_j \in \text{Hom}_{\tilde{K}}(W_{\sigma_j}, V)$  and  $0 \neq w_j \in W_{\sigma_j}$  such that  $0 \neq \varphi_j(w_j) \in V$ . As  $T(\mathcal{P}_i) = V$ , take  $f_j \in \mathcal{P}_i$  such that  $T(f_j) = \varphi_j(w_j)$ . Let  $E_j$  be the sub- $\tilde{K}$ -module of  $\mathcal{P}_i$  generated by  $f_j$ . Clearly,  $T(E_j)$  is the sub- $\tilde{K}$ -module of  $V$  generated by  $\varphi_j(w_j)$ , which indeed is  $\varphi_j(W_{\sigma_j}) \simeq W_{\sigma_j}$  as  $\tilde{K}$ -modules by Schur's lemma. So  $T|_{E_j}$  gives rise to a non-zero element of  $\text{Hom}_{\tilde{K}}(E_j, \sigma_j)$ .

As  $E_j$  is finite-dimensional,  $\text{Hom}_{\tilde{K}}(\sigma_j, E_j) \neq 0$  and  $\sigma_j \in \mathcal{R}(\tilde{K}, E_j) \subseteq \mathcal{R}(\tilde{K}, \mathcal{P}_i)$  for both  $j \in \{0, 1\}$ . By Corollary 2.4,  $\sigma_1$  and  $\sigma_2 \notin \mathcal{R}(\tilde{K}, \mathcal{P}_{1-i})$ . Therefore by definition,  $\deg(\sigma_1) \equiv \deg(\sigma_2) \equiv i \pmod{2}$ .  $\square$

### 3. NON-VANISHING

In this section, we quote some non-vanishing results for theta liftings.

Let  $W$  be a real symplectic vector space. A reductive dual pair  $(G, G')$  in  $Sp(W)$  is called *irreducible* if  $G \cdot G'$  acts irreducibly on  $W$ . Every reductive dual pair  $(G, G')$  in  $Sp(W)$  can be decomposed into a direct sum of irreducible pairs, namely, there is an orthogonal direct sum decomposition  $W = \bigoplus_{i=1}^k W_i$  such that  $G \cdot G'$  acts irreducibly on  $W_i$ , and the restrictions of  $(G, G')$  to  $W_i$  define irreducible reductive dual pairs  $(G_i, G'_i)$  in  $Sp(W_i)$ . Indeed,  $G = G_1 \times \cdots \times G_k$  and  $G' = G'_1 \times \cdots \times G'_k$ . All irreducible real reductive dual pairs are classified as follows ([How79]).

| Type I                                  | $\mathbf{Sp}$                 | Type II                                | $\mathbf{Sp}$         |
|---|-------------------------------|--|-----------------------|
| $(O(p, q), Sp(2n, \mathbb{R}))$         | $Sp(2n(p+q), \mathbb{R})$     | $(GL_m(\mathbb{R}), GL_n(\mathbb{R}))$ | $Sp(2mn, \mathbb{R})$ |
| $(O_p(\mathbb{C}), Sp(2n, \mathbb{C}))$ | $Sp(4pn, \mathbb{R})$         | $(GL_m(\mathbb{C}), GL_n(\mathbb{C}))$ | $Sp(4mn, \mathbb{R})$ |
| $(Sp(p, q), O^*(2n))$                   | $Sp(4n(p+q), \mathbb{R})$     | $(GL_m(\mathbb{H}), GL_n(\mathbb{H}))$ | $Sp(8mn, \mathbb{R})$ |
| $(U(p, q), U(r, s))$                    | $Sp(2(p+q)(r+s), \mathbb{R})$ |  |                       |

An irreducible real reductive dual pair  $(G, G')$  of type I is said to be *in the stable range* with  $G$  the smaller member if the defining module of  $G'$  contains an isotropic subspace of the same dimension as that of the defining module of  $G$ . All real reductive dual pairs  $(G_1, G_2)$  in the stable range are listed as:

| $G_1$             | $G_2$                 | in the stable range |                    |
|-------------------|-----------------------|---------------------|--------------------|
|                   |                       | with $G_1$ smaller  | with $G_2$ smaller |
| $O_p(\mathbb{C})$ | $Sp_{2n}(\mathbb{C})$ | $n \geq p$          | $p \geq 4n$        |
| $O(p, q)$         | $Sp_{2n}(\mathbb{R})$ | $n \geq p+q$        | $p, q \geq 2n$     |
| $Sp(p, q)$        | $O^*(2n)$             | $n \geq 2(p+q)$     | $p, q \geq n$      |
| $U(p, q)$         | $U(r, s)$             | $r, s \geq p+q$     | $p, q \geq r+s$    |

Note that  $\text{Ker}(\widetilde{\mathbf{Sp}} \rightarrow \mathbf{Sp}) = \mu_2 = \{\pm 1\}$ . A representation of  $\widetilde{G}$  is called genuine if it is non-trivial on  $\text{Ker}(\widetilde{G} \rightarrow G) = \mu_2$ . Let  $\mathcal{R}_{gen}(\widetilde{G}) = \{\text{genuine } \pi \in \mathcal{R}(\widetilde{G})\}$ . Clearly,  $\{\pi \in \mathcal{R}(\widetilde{G}) \mid \theta(\pi) \neq 0\} \subseteq \mathcal{R}_{gen}(\widetilde{G})$  since  $\omega$  is genuine.

**Lemma 3.1** (Non-vanishing of theta liftings in the stable range [PP08]). *If  $(G, G')$  is an irreducible real reductive dual pair of type I in the stable range with  $G$  the smaller member, then  $\mathcal{R}_{gen}(\widetilde{G}) = \{\pi \in \mathcal{R}(\widetilde{G}) \mid \theta(\pi) \neq 0\}$ .*

**Lemma 3.2** ([Moeg89, AB95, LPTZ03]). *For  $(G, G') = (GL_m(F), GL_n(F))$  with  $F = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ , if  $n \geq m$ , then  $\mathcal{R}_{gen}(\widetilde{G}) = \{\pi \in \mathcal{R}(\widetilde{G}) \mid \theta(\pi) \neq 0\}$ .*

#### 4. SPLITTING

To pass from  $\widetilde{G}$  to  $G$  for our purpose, we want the covering  $\widetilde{G} \rightarrow G$  to split.

We say the metaplectic covering  $\widetilde{\mathbf{Sp}} \rightarrow \mathbf{Sp}$  splits over  $G$  if the covering  $\widetilde{G} \rightarrow G$  splits, namely, the short exact sequence  $1 \rightarrow \mu_2 \rightarrow \widetilde{G} \rightarrow G \rightarrow 1$  splits. For an irreducible real reductive dual pair  $(G_1, G_2)$ , a sufficient condition ([AB95, AB98, Pau98, Ada07]) for  $\widetilde{\mathbf{Sp}} \rightarrow \mathbf{Sp}$  to split over  $G_i$  is given as follows:

| $G_1$              | $G_2$                 | the metaplectic covering |                   |
|--------------------|-----------------------|--------------------------|-------------------|
|                    |                       | splits over $G_1$        | splits over $G_2$ |
| $O_p(\mathbb{C})$  | $Sp_{2n}(\mathbb{C})$ | always                   |                   |
| $O(p, q)$          | $Sp_{2n}(\mathbb{R})$ | if $n$ is even           | if $p+q$ is even  |
| $Sp(p, q)$         | $O^*(2n)$             | always                   |                   |
| $U(p, q)$          | $U(r, s)$             | if $r+s$ is even         | if $p+q$ is even  |
| $GL_m(\mathbb{R})$ | $GL_n(\mathbb{R})$    | if $n$ is even           | if $m$ is even    |
| $GL_m(\mathbb{C})$ | $GL_n(\mathbb{C})$    | always                   |                   |
| $GL_m(\mathbb{H})$ | $GL_n(\mathbb{H})$    | always                   |                   |

When  $\widetilde{\mathbf{Sp}} \rightarrow \mathbf{Sp}$  splits over  $G$ , we identify  $\widetilde{G}$  with  $G \times \mu_2$ , and  $\widetilde{G} \rightarrow G$  becomes the nature projection  $G \times \mu_2 \rightarrow G$ . Then  $\mathcal{R}_{gen}(\widetilde{G}) = \{\pi \otimes \text{sgn} \mid \pi \in \mathcal{R}(G)\}$ , where  $\text{sgn}$  is the sign character of  $\mu_2$  (which sends  $-1$  to  $-1$ ). Under this identification,  $\widetilde{K} = K \times \mu_2$ , and  $\mathcal{R}_{gen}(\widetilde{K}) = \{\sigma \otimes \text{sgn} \mid \sigma \in \mathcal{R}(K)\}$ . Consider the actions of  $K$  on the Fock model  $\mathcal{P}$  via  $K \cong K \times \{1\} \subset K \times \mu_2 = \widetilde{K}$ , then  $\mathcal{R}(K, \mathcal{P}) = \{\sigma \in \mathcal{R}(K) \mid$

$\sigma \otimes \text{sgn} \in \mathcal{R}(\tilde{K}, \mathcal{P})\}$ . For  $\sigma \in \mathcal{R}(K, \mathcal{P})$ , similarly as before, we define  $\deg(\sigma)$  (with respect to  $(G, G')$ ) as the minimal degree of polynomials in the  $\sigma$ -isotypic subspace  $\mathcal{P}_\sigma = \sum_{\varphi \in \text{Hom}_K(\sigma, \mathcal{P})} \text{Im}(\varphi)$ . Clearly,  $\deg(\sigma) = \deg(\sigma \otimes \text{sgn})$ .

### 5. PROOF OF THEOREM 1.3

**Proposition 5.1.** *Let  $(G, G')$  be an irreducible real reductive dual pair, either in the stable range or of type II, with  $G$  the smaller member, and the metaplectic covering splitting over  $G$ . If  $\pi \in \mathcal{R}(G)$ ,  $\sigma_1, \sigma_2 \in \mathcal{R}(K, \pi)$ , then  $\deg(\sigma_1) \equiv \deg(\sigma_2) \pmod{2}$ .*

*Proof.* By Lemma 3.1, 3.2,  $\theta(\pi \otimes \text{sgn}) \neq 0$ . For  $i \in \{0, 1\}$ ,  $\sigma_i \otimes \text{sgn} \in \mathcal{R}(\tilde{K}, \pi \otimes \text{sgn})$ . By Theorem 2.5,  $\deg(\sigma_1) \equiv \deg(\sigma_1 \otimes \text{sgn}) \equiv \deg(\sigma_2 \otimes \text{sgn}) \equiv \deg(\sigma_2) \pmod{2}$ .  $\square$

To prove Theorem 1.3 from Proposition 5.1, it suffices to find a suitable  $G'$  such that  $(G, G')$  is an irreducible real reductive dual pair satisfying three conditions:

- (1) It is either in the stable range or of type II, with  $G$  the smaller member.
- (2) The metaplectic covering  $\tilde{\mathbf{Sp}} \rightarrow \mathbf{Sp}$  splits over  $G$ .
- (3) With respect to this dual pair,  $\varepsilon(\sigma) \equiv \deg(\sigma) \pmod{2}$  for any  $\sigma \in \mathcal{R}(K, \mathcal{P})$ .

We can take  $G'$  according to the following table, where conditions (1), (2) and (3) are translated into explicit requirements. In this table,  $F = \mathbb{R}, \mathbb{C}$ , or  $\mathbb{H}$ , and the column for condition (3) is obtained from the following Lemma 5.2.

| $G$                   | $G'$                  | condition (1)     | condition (2)  | condition (3)  |
|-----------------------|-----------------------|-------------------|----------------|----------------|
| $O_p(\mathbb{C})$     | $Sp_{2n}(\mathbb{C})$ | $n \geq p$        |                |                |
| $Sp_{2n}(\mathbb{C})$ | $O_p(\mathbb{C})$     | $p \geq 4n$       |                |                |
| $O(p, q)$             | $Sp_{2n}(\mathbb{R})$ | $n \geq p + q$    | $2 \mid n$     |                |
| $Sp_{2n}(\mathbb{R})$ | $O(p, q)$             | $p, q \geq 2n$    | $2 \mid p + q$ | $4 \mid p - q$ |
| $Sp(p, q)$            | $O^*(2n)$             | $n \geq 2(p + q)$ |                |                |
| $O^*(2n)$             | $Sp(p, q)$            | $p, q \geq n$     |                | $2 \mid p - q$ |
| $U(p, q)$             | $U(r, s)$             | $r, s \geq p + q$ | $2 \mid r + s$ | $4 \mid r - s$ |
| $GL_m(F)$             | $GL_n(F)$             | $n \geq m$        | $2 \mid n$     |                |

**Lemma 5.2** ([Møeg89, AB95, Pau98, LPTZ03, Pau05]). *Let  $(G, G')$  be an irreducible real reductive dual pair such that the metaplectic covering splits over  $G$ . If  $\sigma \in \mathcal{R}(K, \mathcal{P})$ , then  $\deg(\sigma)$  is expressed as in the following table.*

| $G$                   | $G'$                  | $K$                       | $\sigma \in \mathcal{R}(K, \mathcal{P})$   | $\deg(\sigma)$   |
|-----------------------|-----------------------|---------------------------|--|--|
| $O_p(\mathbb{C})$     | $Sp_{2n}(\mathbb{C})$ | $O(p)$                    | $(a_1, \dots, a_{\lfloor \frac{p}{2} \rfloor}; \epsilon)$  | $\sum_{i=1}^{\lfloor \frac{p}{2} \rfloor} a_i + \frac{1-\epsilon}{2}(p - 2\#\{i \mid a_i > 0\})$   |
| $Sp_{2n}(\mathbb{C})$ | $O_p(\mathbb{C})$     | $Sp(n)$                   | $(a_1, \dots, a_n)$  | $\sum_{i=1}^n a_i$   |
| $O(p, q)$             | $Sp_{2n}(\mathbb{R})$ | $O(p)$<br>$\times O(q)$   | $(a_1, \dots, a_{\lfloor \frac{p}{2} \rfloor}; \epsilon)$<br>$\otimes (b_1, \dots, b_{\lfloor \frac{q}{2} \rfloor}; \eta)$ | $\sum_{i=1}^{\lfloor \frac{p}{2} \rfloor} a_i + \frac{1-\epsilon}{2}(p - 2\#\{i \mid a_i > 0\})$<br>$+ \sum_{j=1}^{\lfloor \frac{q}{2} \rfloor} b_j + \frac{1-\eta}{2}(q - 2\#\{j \mid b_j > 0\})$ |
| $Sp_{2n}(\mathbb{R})$ | $O(p, q)$             | $U(n)$                    | $(a_1, \dots, a_n)$  | $\sum_{i=1}^n  a_i - \frac{p-q}{2} $   |
| $Sp(p, q)$            | $O^*(2n)$             | $Sp(p)$<br>$\times Sp(q)$ | $(a_1, \dots, a_p)$<br>$\otimes (b_1, \dots, b_q)$   | $\sum_{i=1}^p a_i + \sum_{j=1}^q b_j$  |
| $O^*(2n)$             | $Sp(p, q)$            | $U(n)$                    | $(a_1, \dots, a_n)$  | $\sum_{i=1}^n  a_i - p + q $   |
| $U(p, q)$             | $U(r, s)$             | $U(p)$<br>$\times U(q)$   | $(a_1, \dots, a_p)$<br>$\otimes (b_1, \dots, b_q)$   | $\sum_{i=1}^p  a_i - \frac{r-s}{2} $<br>$+ \sum_{j=1}^q  b_j + \frac{r-s}{2} $   |
| $GL_m(\mathbb{R})$    | $GL_n(\mathbb{R})$    | $O(m)$                    | $(a_1, \dots, a_{\lfloor \frac{m}{2} \rfloor}; \epsilon)$  | $\sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} a_i + \frac{1-\epsilon}{2}(m - 2\#\{i \mid a_i > 0\})$   |
| $GL_m(\mathbb{C})$    | $GL_n(\mathbb{C})$    | $U(m)$                    | $(a_1, \dots, a_m)$  | $\sum_{i=1}^m  a_i $   |
| $GL_m(\mathbb{H})$    | $GL_n(\mathbb{H})$    | $Sp(m)$                   | $(a_1, \dots, a_m)$  | $\sum_{i=1}^m a_i$   |

## 6. GENERALIZATION

Theorem 1.3 can be easily generalized to members of (reducible) real reductive dual pairs.

**Theorem 6.1.** *Let  $G$  be a member of a real reductive dual pair, with a maximal compact subgroup  $K$ . If  $\pi \in \mathcal{R}(G)$  and  $\sigma, \sigma' \in \mathcal{R}(K, \pi)$ , then  $\varepsilon(\sigma) = \varepsilon(\sigma')$ .*

*Proof.* Any real reductive dual pair is a direct sum of irreducible ones, so  $G = G_1 \times G_2 \times \cdots \times G_r$  with each  $G_i$  a member of an irreducible real reductive dual pair. Then  $K = K_1 \times \cdots \times K_r$  with  $K_i$  a maximal compact subgroup of  $G_i$ . By [GK13],  $\pi = \bigotimes_{i=1}^r \pi_i$  with  $\pi_i \in \mathcal{R}(G_i)$ . Moreover,  $\sigma = \bigotimes_{i=1}^r \sigma_i$  and  $\sigma' = \bigotimes_{i=1}^r \sigma'_i$ , with  $\sigma_i$  and  $\sigma'_i \in \mathcal{R}(K_i, \pi_i)$ . Theorem 1.3 holds for  $(G_i, K_i)$  (up to isomorphisms). So  $\varepsilon(\sigma_i) = \varepsilon(\sigma'_i)$  for all  $i$ . Therefore,  $\varepsilon(\sigma) = \sum_{i=1}^r \varepsilon(\sigma_i) = \sum_{i=1}^r \varepsilon(\sigma'_i) = \varepsilon(\sigma')$ .  $\square$

## REFERENCES

- [AB95] Jeffrey Adams and Dan Barbasch, *Reductive dual pair correspondence for complex groups*, J. Funct. Anal. **132** (1995), no. 1, 1–42. MR 1346217 (96h:22003)
- [AB98] ———, *Genuine representations of the metaplectic group*, Compositio Math. **113** (1998), no. 1, 23–66. MR 1638210 (99h:22013)
- [Ada07] Jeffrey Adams, *The theta correspondence over  $\mathbb{R}$* , Harmonic analysis, group representations, automorphic forms and invariant theory, Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap., vol. 12, World Sci. Publ., Hackensack, NJ, 2007, pp. 1–39. MR 2401808
- [GK13] Dmitry Gourevitch and Alexander Kemarsky, *Irreducible representations of a product of real reductive groups*, J. Lie Theory **23** (2013), no. 4, 1005–1010. MR 3185208
- [How79] Roger Howe,  *$\theta$ -series and invariant theory*, Automorphic forms, representations and  $L$ -functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 275–285. MR 546602 (81f:22034)
- [How89] ———, *Transcending classical invariant theory*, J. Amer. Math. Soc. **2** (1989), no. 3, 535–552. MR 985172 (90k:22016)
- [LPTZ03] Jian-Shu Li, Annegret Paul, Eng-Chye Tan, and Chen-Bo Zhu, *The explicit duality correspondence of  $(Sp(p, q), O^*(2n))$* , J. Funct. Anal. **200** (2003), no. 1, 71–100. MR 1974089 (2004c:22018)
- [LV80] Gérard Lion and Michèle Vergne, *The Weil representation, Maslov index and Theta series*, Progress in Mathematics, vol. 6, Birkhäuser, Boston, Mass., 1980. MR 573448 (81j:58075)
- [Moeg89] Colette Mœglin, *Correspondance de Howe pour les paires reductives duales: quelques calculs dans le cas archimédien*, J. Funct. Anal. **85** (1989), no. 1, 1–85. MR 1005856 (91b:22021)
- [Pau98] Annegret Paul, *Howe correspondence for real unitary groups*, J. Funct. Anal. **159** (1998), no. 2, 384–431. MR 1658091 (2000m:22016)
- [Pau05] ———, *On the Howe correspondence for symplectic-orthogonal dual pairs*, J. Funct. Anal. **228** (2005), no. 2, 270–310. MR 2175409 (2006g:20076)
- [PP08] Victor Protsak and Tomasz Przebinda, *On the occurrence of admissible representations in the real Howe correspondence in stable range*, Manuscripta Math. **126** (2008), no. 2, 135–141. MR 2403182 (2008m:22022)
- [Sha62] David Shale, *Linear symmetries of free boson fields*, Trans. Amer. Math. Soc. **103** (1962), 149–167. MR 0137504
- [Wei64] André Weil, *Sur certains groupes d'opérateurs unitaires*, Acta Math. **111** (1964), 143–211. MR 0165033
- [Wey39] Hermann Weyl, *The classical groups: their invariants and representations*, Princeton University Press, Princeton, N.J., 1939. MR 0000255 (1,42c)